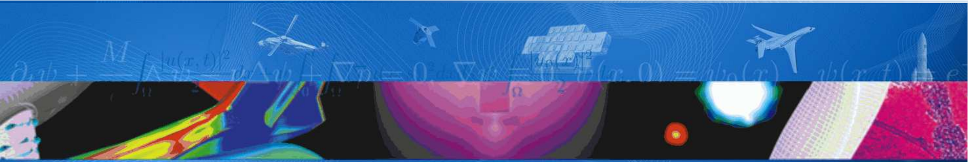

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Modelling the Interaction of Stochastic Electromagnetic Fields with Stochastic Structures

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Outline

This talk has three parts:

- Part 1: Introduction to stochastic interactions
- Part 2: Observables on stochastic EM fields
- Part 3: Integral equations on stochastic surfaces

Part I

Introduction to stochastic interactions

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Lossy models need stochastic completion

Here is an interesting conflict:

- Theoretical physics is based on "energy conservation"
- Engineering models account for energy loss

We conclude that such engineering models *are essentially incomplete.*

- If we admit energy loss in our models we implicitly admit that our models account for phenomena which we cannot model precisely
- From general reciprocity principles, we conclude that there should also be mechanisms which provide energy to our system.
- As we do not model these mechanisms in detail we have to complete our lossy-model by adding stochastic sources.

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Stochastic model completion

We come to the conclusion that:

Any model which has loss-coefficients should also have stochastic source-terms.

This is most familiar in electronics, where noise sources are added to resistors. We have antenna noise but the relation with the radiation resistance is already less well-known.
In general:

Any practical deterministic model is necessarily incomplete and, hence, requires "stochastic completion."

We are looking for methods to make such completions.

Observable models

A general model gets this form

$$\psi = H(\psi')$$

Here, ψ is a response, ψ' is a control and H is some mapping corresponding to the relation between controls and responses. In the linear (affine) case:

$$\psi = F + H\psi'$$

F is a response which is not-controlled by the model controls. This is an observable model, the coefficients of F and H are observable as the ψ and ψ' are.

Linear observables through distributions

The proto-type observables of electromagnetism can be expressed as the evaluation of a vector-valued distribution on an electromagnetic field.

$$V = \langle j, E \rangle$$
$$I = \langle k, H \rangle$$

where $\{E, H\}$ are vector valued functions on space-time representing the electromagnetic field and j and k are distributions. These expressions provide the link between field theory and observable models.

The variance of observables

Observables are defined through distributions on EM fields

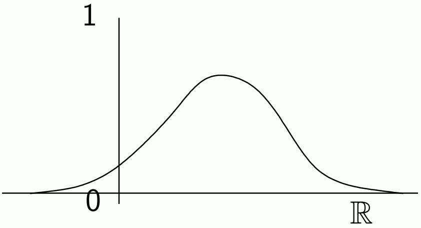
$$V = \langle j, E \rangle$$

If the field is stochastic, the values of observables are stochastic.
We can relate the statistics of an observable to the stochastic field.
Supposing $\mathbb{E}[E] = 0$,

$$\begin{aligned} \mathbb{E}[V] &= \langle j, \mathbb{E}[E] \rangle = 0 \\ \mathrm{var}[V] = \mathbb{E}[|V|^2] &= \mathbb{E}\left[\langle j, E \rangle \overline{\langle j, E \rangle}\right] \\ &= \langle j, C_E \bar{j} \rangle \end{aligned}$$

where C_E is the covariance operator of the stochastic field.

Stochastic variables



- A (real) stochastic variable is a probability distribution on the real numbers.
- A single precise value is replaced by (precise!) statistics (like average, variance, etc.)
- In this way, one models the uncertain outcome of measurements, computations etc.

Stochastic fields

- A stochastic field on a given domain is a function associating stochastic variables to the points of the domain.
- A stochastic electromagnetic field is a stochastic field "satisfying the Maxwell equations."

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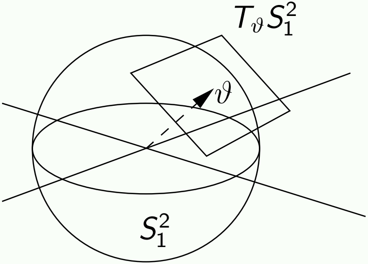
- How can a stochastic field satisfy the Maxwell equations ??

Stochastic fields

- A stochastic field on a given domain is a function associating stochastic variables to the points of the domain.
- A stochastic electromagnetic field is a stochastic field "satisfying the Maxwell equations."
- How can a stochastic field satisfy the Maxwell equations ??
- By analogy with a stochastic variable, we define a stochastic EM field through a probability measure on the space of solutions of (BVPs for) the Maxwell equations.

Example: A stochastic plane wave



$$E(x) = e_T \exp(-ik\vartheta \cdot x)$$

with $e_T \in \mathbb{R}$ and $e_T \cdot \vartheta = 0$. Putting a probability distribution on the (complexified) TS_1^2 with unit variance on each tangent space, we get a covariance function

$$c(x,y) = (4\pi)(\mathbb{I} + k_0^{-2}\nabla\nabla^t) \frac{\sin(k_0\|x-y\|)}{k_0\|x-y\|}$$

which will be seen to be proportional to the kernel of the radiation-loss operator.

Spectral density

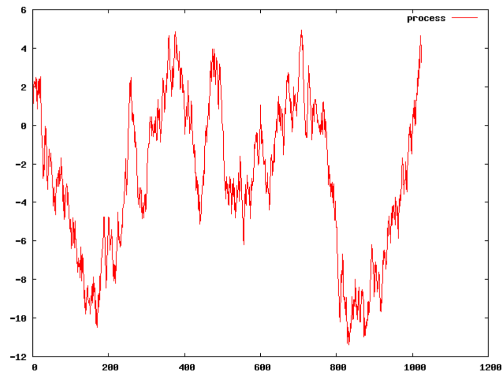
If we define a stochastic process by means of a Fourier integral representation

$$A(t) = \int_{\omega \in \mathbb{R}} \exp(-j\omega t)\widehat{A}(\omega)d\omega$$

where $\mathbb{E}[A(\omega)\overline{A(\nu)}] = \delta(\omega - \nu)R(\omega)$, we get a process with the correct auto-correlation function.

Interpretation questions

The question whether a specific realisation of a stochastic process is *representative* for what can be actually observed *has no answer*.



What we can check is whether certain interesting statistics converge to the ones obtained on the actual phenomena.

Observables defined by a scattering problem I

Most observables can be interpreted as interaction coefficients and the current distribution defining them can be defined as the solution of a scattering problem.

Simplest example: Scattering by a perfect conductor: a plane-wave scattering coefficient is defined by the current distribution, induced by the incident wave on the obstacle, evaluated on a time-reversed plane wave for the observation direction and polarisation.

Such current distributions can be constructed by solving the boundary value problems numerically.

Observables defined by a scattering problem II

Here again uncertainty pops-up.

- The geometrical model we use in the BVP may not correspond exactly with the actual configuration
- The constitutive coefficients we use in the Maxwell equations may reflect only approximately the materials which are actually used

If we look for observable variances of the combination of stochastic distributions evaluated on stochastic fields we are led to

$$\mathrm{var}(V) = \mathrm{Tr}(C_J C_E)$$

where $\mathrm{Tr}(X)$ is the trace of operator X. In fact, we show that the canonical stochastic fields, which we define in Part II, allow for the evaluation of the covariance of observables defined by the stochastic surface distributions we define in Part III.

Towards a general theory

To sum up:

- Observables are the quantities of interest in EM modelling,
- We have seen what a stochastic electromagnetic field can be
- An important rôle is played by the field's covariance operator in computing the elementary statistics of observables
- We would like to obtain a stochastic field with a "natural" covariance operator depending on the geometrical and physical configuration in which it is defined.
- We want to model the uncertainty in the definition of the observable itself

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Part II

Observables on stochastic EM fields

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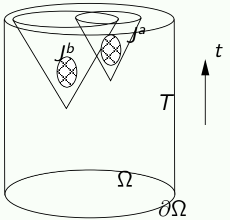
Integral relations I

A fundamental global relation

$$\int_T \int_{\partial\Omega} n \cdot (E^a \times H^b + E^b \times H^a) = -\int_{\partial T} \int_\Omega (\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b) + \int_T \int_\Omega (E^a \cdot J^b + E^b \cdot J^a)$$



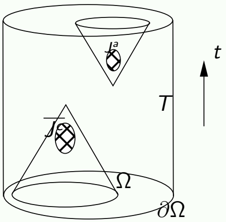
By taking Ω sufficiently large, the left hand side vanishes and if T contains the source durations, the causal fields only differ from zero on the upper limit of $T = (t_0, t_1)$.

$$\int_{t_1 \times \Omega} (\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b) = \int_{T \times \Omega} (E^a \cdot J^b + E^b \cdot J^a) \qquad (1)$$

Integral relations II (Reciprocity)

Time reversal for the field $\{E^b, H^b, J^b\} = \{\overline{E^c}, \overline{H^c}, \overline{J^c}\}$:

$$\int_T \int_{\partial\Omega} n \cdot (E^a \times \overline{H^c} + \overline{E^c} \times H^a) = -\int_{\partial T} \int_{\Omega} (\mu_0 H^a \cdot \overline{H^c} + \varepsilon_0 E^a \cdot \overline{E^c}) + \int_T \int_{\Omega} (E^a \cdot \overline{J^c} + \overline{E^c} \cdot J^a)$$



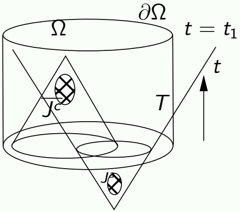
By taking Ω and T sufficiently large, the left hand side vanishes. The first term on the right hand side vanishes too because on the extreme times either the causal or the anti-causal field vanishes.

$$\int_T \int_{\Omega} (E^a \cdot \overline{J^c} + \overline{E^c} \cdot J^a) = 0 \qquad (2)$$

Integral relations III

Choosing a different situation for the mixed causal/anti-causal case:

$$\int_T \int_{\partial\Omega} n \cdot (E^a \times \overline{H^c} + \overline{E^c} \times H^a) = -\int_{\partial T} \int_{\Omega} (\mu_0 H^a \cdot \overline{H^c} + \varepsilon_0 E^a \cdot \overline{E^c}) + \int_T \int_{\Omega} (E^a \cdot \overline{J^c} + \overline{E^c} \cdot J^a)$$



By taking Ω and T sufficiently large, the left hand side vanishes. The first term on the right hand side vanishes in $t = t_1$.

$$\int_{t_0 \times \Omega} (\mu_0 H^a \cdot \overline{H^c} + \varepsilon_0 E^a \cdot \overline{E^c}) = \int_T \int_{\Omega} (E^a \cdot \overline{J^c}) \qquad (3)$$

On a well-chosen space-time domain, we have equations (1) and (2)

$$\int_T \int_\Omega (E^a \cdot J^b + E^b \cdot J^a) = \int_{t_1 \times \Omega} (\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b)$$

$$\int_T \int_\Omega (E^a \cdot \overline{J^c} + \overline{E^c} \cdot J^a) = 0$$

Energy emission correlation

On a well-chosen space-time domain, we have equations (1) and (2)

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$$\int_T \int_\Omega (E^a \cdot \overline{J^c} + \overline{E^c} \cdot J^a) = 0$$

Writing $J^b = \overline{J^c}$ and hence $\overline{E^c} = E^{ac}_{J^b}$ (anti-causal field of J^b):

$$\int_T \int_\Omega (E^a \cdot J^b + E^{ac}_{J^b} \cdot J^a) = 0$$

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Energy emission correlation

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Energy emission correlation

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$$\int_T \int_\Omega (E^a \cdot J^b + E^b \cdot J^a) = \int_{t_1 \times \Omega} (\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b)$$

$$\int_T \int_\Omega (E^a \cdot J^b + E^{ac}_{J^b} \cdot J^a) = 0$$

Subtraction gives

$$\int_{T \times \Omega} J^a \cdot (E^b - E^{ac;b}) = \int_{T \times \Omega} J^a \cdot C(J^b) = \int_{\{t_1\} \times \Omega} (\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b) \tag{4}$$

Here $C : J \mapsto E_J - E^{ac}_J$ is seen to be an operator measuring emitted energy.

Contents

Canonical stochastic fields

The right hand side of the last equation defines a metric on a Hilbert space H of vector fields on space ($\approx H(\mathrm{curl}, \Omega)$).

$$\int_T \int_\Omega J^a \cdot C(J^b) = (E^a, E^b)_H$$

Now, we consider a "Gel'fand triple":

$$S \subset H \subset S'$$

Canonical stochastic fields

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Now, we Smooth functions nd triple": Tempered distributions

$$S \subset H \subset S'$$

Canonical stochastic fields

The right hand side of the last equation defines a metric on a Hilbert space H of vector fields on space ($\approx H(\mathrm{curl}, \Omega)$).

$$\int_T \int_\Omega J^a \cdot C(J^b) = (E^a, E^b)_H$$

Now, we consider a "Gel'fand triple":

$$S \subset H \subset S'$$

Following a theorem by Minlos, we can define a unique probability measure on S', such that its covariance operator equals the identity on H. This implies the existence of a stochastic distribution e_0 such that

$$(\forall f, g \in S) \quad \mathbb{E}[\langle e_0, f\rangle\langle e_0, g\rangle] = (f, g)_H$$

Canonical stochastic fields II

Starting from the last equation, $\mathbb{E}[\langle e_0, f\rangle\langle e_0, g\rangle] = (f, g)_H$, using time-reversal on (4),

Canonical stochastic fields II

Starting from the la $\int_{\{t_1\}\times\Omega}(\mu_0 H^a \cdot H^b + \varepsilon_0 E^a \cdot E^b) = \int_{T\times\Omega} J^a \cdot C(J^b)$

time-reversal on (4), gives

$$\int_{\{t_0\}\times\Omega}(\mu_0 \overline{H^a} \cdot \overline{H^b} + \varepsilon_0 \overline{E^a} \cdot \overline{E^b}) = \int_{T\times\Omega} \overline{J^a} \cdot C(\overline{J^b})$$

Canonical stochastic fields II

Starting from the last equation, $\mathbb{E}[\langle e_0, f\rangle\langle e_0, g\rangle] = (f, g)_H$, using time-reversal on (4), we get

$$\mathbb{E}[\langle e_0, \overline{f^a}\rangle\langle e_0, \overline{g^b}\rangle] = \int_{T\times\Omega} \overline{J^a} \cdot C(\overline{J^b})$$

together with (3),

$$\int_{t_0\times\Omega} (\mu_0 H^a \cdot \overline{H^c} + \varepsilon_0 E^a \cdot \overline{E^c}) = \int_T \int_\Omega (E^a \cdot \overline{J^c})$$

Canonical stochastic fields II

Starting from the last equation, $\mathbb{E}[\langle e_0, f\rangle\langle e_0, g\rangle] = (f, g)_H$, using time-reversal on (4), we get

$$\mathbb{E}[\langle e_0, \overline{f^a}\rangle\langle e_0, \overline{g^b}\rangle] = \int_{T\times\Omega} \overline{J^a} \cdot C(\overline{J^b})$$

together with (3) generalised to distributions,

$$\langle e_0, \overline{f^c}\rangle = (\int_{t_0\times\Omega} (\mu_0 H_0 \cdot \overline{H^c} + \varepsilon_0 E_0 \cdot \overline{E^c}) \quad) = \int_T \int_\Omega (E_0 \cdot \overline{J^c})$$

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Canonical stochastic fields II

Starting from the last equation, $\mathbb{E}[\langle e_0, f\rangle\langle e_0, g\rangle] = (f, g)_H$, using time-reversal on (4), we obtain

$$\mathbb{E}\Big[\int_{\Omega\times T}(E_0 \cdot \overline{J^a}) \int_{\Omega\times T}(E_0 \cdot \overline{J^b})\ \Big] = \langle\overline{J^a}, C(\overline{J^b})\rangle_{\Omega\times T}$$

and indeed

$$\langle\overline{J^a}, C_{E_0}(\overline{J^b})\rangle_{\Omega\times T} = \langle\overline{J^a}, C(\overline{J^b})\rangle_{\Omega\times T}$$

where E_0 is the distributional electric field corresponding to the stochastic initial value distribution e_0.
Because, J^a and J^b are arbitrary, this shows the equivalence of the covariance operator of the stochastic field E_0, i.e., C_{E_0}, and the energy emission operator C.

Contents

Properties of the covariance operator I

- The analysis, so far, applies to smooth current distributions evaluated on non-smooth stochastic field distributions.
- However, many observables are defined by surface current distributions computed from boundary value problems.
- The evaluation of a distribution from $S'(\mathbb{R}^3)$ on a current distribution with support on a surface $\Gamma \subset \mathbb{R}^3$ is not defined.
- Perhaps, in spite of the above, we can still evaluate the covariance operator?

Properties of the covariance operator II

The essential properties of the space-time covariance operator are those of the following operator, working on a field $\phi : \mathbb{R}^4 \to \mathbb{R}$,

$$K(\phi)(x,t) = \mathrm{Pv} \int_{\mathbb{R}^3} \frac{1}{4\pi R_x} \tau^*_{(x,t)} \phi$$

where

$$R_x : \mathbb{R}^3 \ni y \mapsto \{\sum_{k=1}^{3} (x^k - y^k)^2\}^{1/2}$$
$$\tau_{(x,t)} : \mathbb{R}^3 \ni y \mapsto (y, t - R_x(y)/c) \in \mathbb{R}^4$$

For any given t, this defines a conventional potential operator on $\mathbb{R}^3$ on the projection of ϕ along the time coordinate.

Properties of the covariance operator III

The covariance operator of a canonical stochastic electromagnetic field is continuous

$$K : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$$

The boundary distributions, j, of electromagnetic fields are (instantaneously) elements of $H^{-\frac{1}{2}}(\delta, \Gamma) = \gamma[H(\delta, \mathbb{R}^3)]$. Therefore,

$$K : H^{-1}(d, \mathbb{R}^3) \ni j \mapsto K(j) \in H^1(d, \mathbb{R}^3)$$

The dominant part of the operator C is the differential of K

$$dK : H^{-1}(d, \mathbb{R}^3) \ni j \mapsto dK(j) \in H(d, \mathbb{R}^3) \text{ and } H^{-\frac{1}{2}}(d, \Gamma) = \gamma'[H(d, \mathbb{R}^3)]$$

We can conclude that, although canonical stochastic fields are singular distributions, we can compute the auto-correlation function of observables defined by surface distributions.

Relation with the radiation resistance I

To relate auto-covariance and radiation resistances, we start with a spectral decomposition

$$V(t) = \int_{\omega \in \mathbb{R}} V_\omega(t)$$

with $V_\omega(t) = \widehat{V}(\omega)\exp(j\omega t)$ and $\widehat{V}(-\omega) = \overline{\widehat{V}(\omega)}$

$$V_\omega(t) = \langle \overline{j_{(\omega,t)}}, E_0 \rangle = \langle \overline{j_\omega}, E_0 \rangle \exp(j\omega t)$$

where

$$\overline{j_\omega}(x, t') = \widehat{j}(x, \omega)\exp(-j\omega t')$$

is the (time-reversed) time-harmonic spectral component of the current distribution defining the observable.

Relation with the radiation resistance II

Now, we define the autocorrelation of the observable V

$$
\begin{aligned}
R_V(s,t) &= \mathbb{E}[V(s)V(t)] \\
&= \mathbb{E}[\int_{\omega\in\mathbb{R}} V_\omega(s) \int_{\nu\in\mathbb{R}} V_\nu(t)] \\
&= \int_{\omega,\nu\in\mathbb{R}^2} \mathbb{E}[V_\omega(s)V_\nu(t)] \\
&= \int_{\omega,\nu\in\mathbb{R}^2} \langle j_\omega(s), Cj_\nu(t)\rangle
\end{aligned}
$$

The canonical stochastic fields we are using here are stationary in the sense that at any given time we have the same spatial stochastic field and that the auto-covariance operator is invariant under time translations.

Relation with the radiation resistance III

This allows us to factor out an offset phase factor

$$\forall t \in \mathbb{R} \quad C(t,s) = c(s-t) = \int_{\omega,\nu \in \mathbb{R}^2} \exp(j\omega t)\exp(j\nu s) f(\omega,\nu)$$

$$\forall t \in \mathbb{R} \quad c(\tau) = \int_{\omega,\nu \in \mathbb{R}^2} \exp(j(\omega+\nu)t)\exp(j\nu\tau) f(\omega,\nu)$$

Therefore, we can take the limit $t \to \infty$ in the RHS

$$c(\tau) = \int_{\omega \in \mathbb{R}} \exp(-j\omega\tau) f(\omega,-\omega)$$

Therewith, we get the auto-covariance function of the observable process

$$C_V(\tau) = \int_{\omega \in \mathbb{R}} (\langle j_\omega, E(j_\omega)\rangle) = \int_{\omega \in \mathbb{R}} R_j(\omega)\exp(j\omega\tau)$$

and $R_j(\omega)$ is the radiation resistance of j_ω.

Contents

Conclusion

- The existence of a stochastic electromagnetic field with a natural covariance operator has been shown (it is a distributional field with “white-noise” initial values),
- This gives a meaning to the computation of ‘a priori’ auto-covariances of stochastic processes induced in systems,
- Realisations of such processes can be used to test systems, not the time-series are to be considered but the statistics we are focusing on.

Part III

Stochastic Boundary Integral Equations

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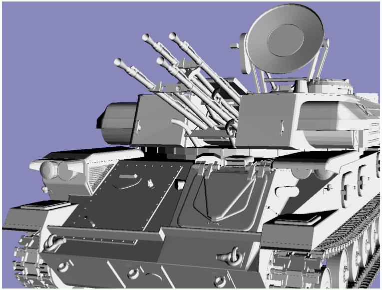
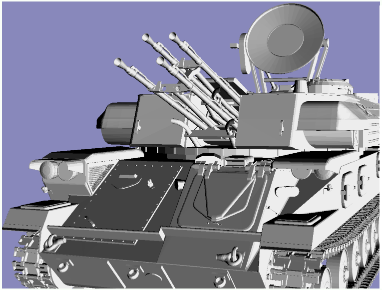
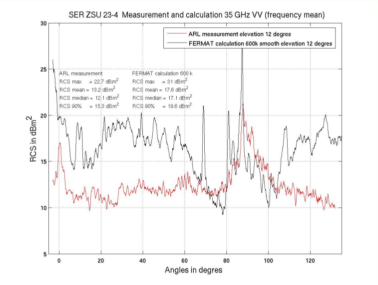
Motivation



We need to compute the scattering diagram of a tank. After spending some time (and some money), we got a jolly nice geometric model, Wow!

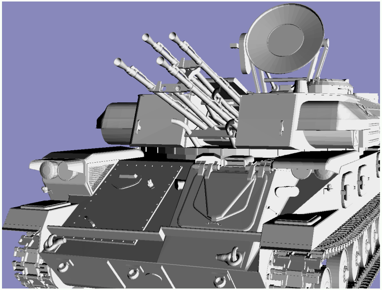


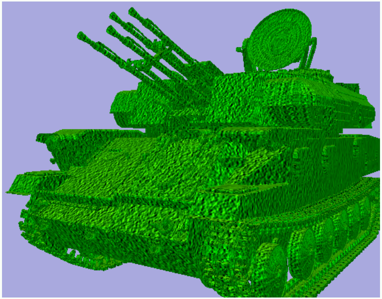


Comparing the computations with the measurements. . . Oops!

The computation are correct on only a few exceptional points.

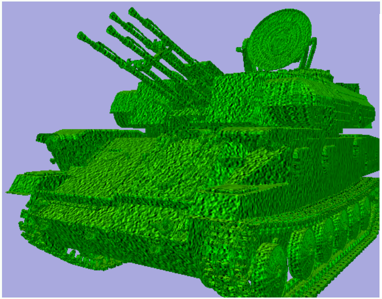
Motivation

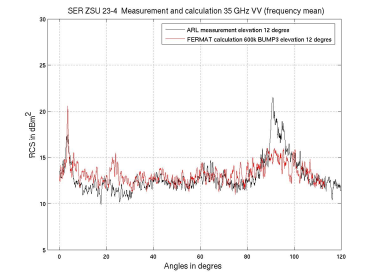




With pain in our heart, we de-jollify our model. . .

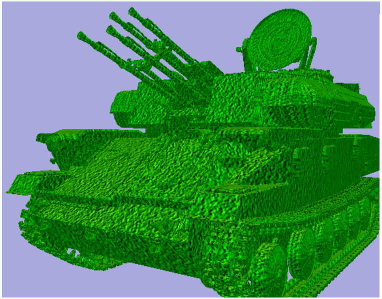
Motivation

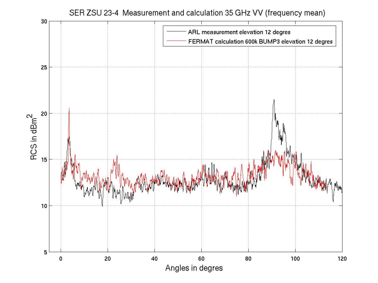




After some trial and error we get realistic results almost everywhere. . .

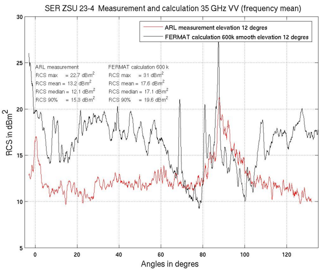
Motivation

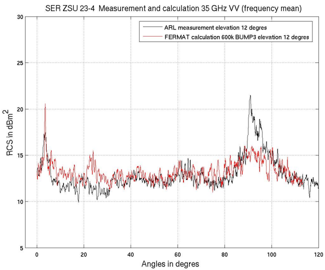




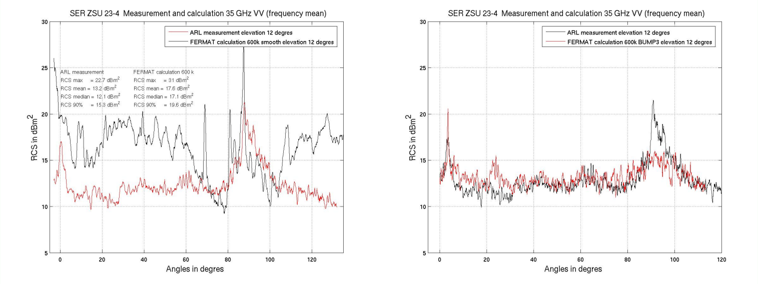
After some trial and error we get realistic results almost everywhere. . . except where we want it!

Motivation


SER ZSU 23-4 Measurement and calculation 35 GHz VV (frequency mean)
ARL measurement elevation 12 degres
FERMAT calculation 600k smooth elevation 12 degres
ARL measurement
RCS max = 22.7 dBm2
RCS mean = 13.2 dBm2
RCS median = 12.1 dBm2
RCS 90% = 15.3 dBm2
FERMAT calculation 600 k
RCS max = 31 dBm2
RCS mean = 17.6 dBm2
RCS median = 17.1 dBm2
RCS 90% = 19.6 dBm2
RCS in dBm2
Angles in degrees

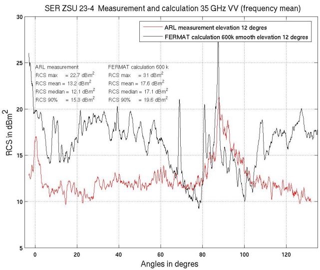

SER ZSU 23-4 Measurement and calculation 35 GHz VV (frequency mean)
ARL measurement elevation 12 degres
FERMAT calculation 600k BUMP3 elevation 12 degres
RCS in dBm2
Angles in degrees

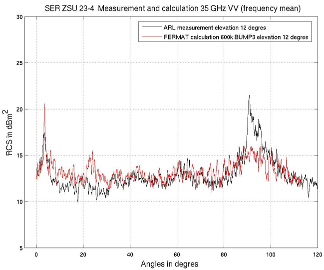
Motivation



As we can see the bumpy model of a tank gives a more realistic impression of the scattering properties. However, the essential features are not all there . . .

Motivation





The message is: we do need precise models. But we need precision in the probability distribution of a stochastic model and not in a single deterministic configuration.

Contents

Plane-wave scattering operator

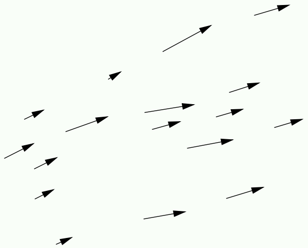
The plane wave scattering operator maps incident plane-waves to far-scattered-fields

$$e^{+}(\vartheta^{+}) = \int_{\vartheta^{-} \in S_1^2} S(\vartheta^{+}, \vartheta^{-}) e^{-}(\vartheta^{-})$$

The scattering coefficients $S(\vartheta^{+}, \vartheta^{-})$ are vector-valued linear forms on the tangent spaces of the unit sphere. They have the following integral representation

$$S_{pq}(\vartheta^{+}, \vartheta^{-}) = j\omega\mu_0 \int_{x \in \partial\Omega} \psi_{-\vartheta^{+}}(x) e_p(\vartheta^{+}) \wedge H_{e_q(\vartheta^{-})}(x)$$

where $e_p(\vartheta)$ is a basis polarisation of a co-tangent frame in (ϑ), $\psi_\vartheta(x) = \exp(-jk\vartheta \cdot x)$ and $H_{e_q(\vartheta^{-})}$ is the surface current density induced by an incident plane wave defined by (e_q, ϑ^{-}).

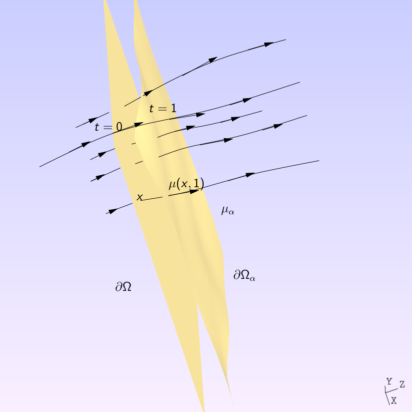


A stochastic linear combination of vector fields

$$v_\alpha = \sum_p \alpha_p v_p$$

with α_p centred random reals. The flow μ_α of the vector field is defined by

$$\partial_t \mu_\alpha(x, t) = v_\alpha(\mu_\alpha(x, t))$$



A stochastic surface as a deformation of an average surface

$$\forall y \in \partial\Omega_\alpha \exists ! x \in \partial\Omega_0 \quad y = \mu_\alpha(x, 1)$$

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Stochastic operator equation

We consider the EFIE on deformed boundaries

$$\Big[\int_{y\in\partial\Omega_\alpha} G_x^{eh}(y)\wedge H(y)\Big]_{\partial\Omega_\alpha} = -\big[E^i\big]_{\partial\Omega_\alpha}(x)$$

transferred to the nominal boundary. First, the integral on the LHS

$$\int_{y\in\partial\Omega_\alpha} G_x^{eh}(y)\wedge H(y) = \int_{y\in\partial\Omega} G_x^{eh}(\mu_\alpha(y))\wedge(\mu_\alpha^* H)(y)$$

and, finally, the equation itself

$$\forall x\in\partial\Omega \int_{y\in\partial\Omega}(\mu_\alpha^*\times\mu_\alpha^* G^{eh})(x,y)\wedge\mu_\alpha^* H(y) = -(\mu_\alpha^* E^i)(x)$$

Written formally as $Bj = e$, with B a stochastic operator and e a stochastic field on $\partial\Omega$.

First order asymptotics I

We call j^n a consistent n-th order asymptotic solution of the integral equation if

$$(B_0 + B_1\alpha + B_2\alpha^2 \dots)(j_0 + j_1\alpha + j_2\alpha^2 \dots) \\ = e_0 + e_1\alpha + e_2\alpha^2 \dots$$

is satisfied for the separate degrees in α up to the degree n. This means that $j^1 = j_0 + j_1\alpha$ is a consistent first-order asymptotic solution if

$$B_0 j_0 = e_0$$
$$B_0 j_1 = e_1 - B_1 j_0$$

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Variances of the scattering coefficients

A plane-wave scattering coefficient as defined before,

$$S_\alpha = \int_{\partial\Omega_\alpha} E \wedge H = \int_{\partial\Omega} \mu_\alpha^* E \wedge \mu_\alpha^* H$$

is a stochastic complex variable. Substitution of the solution of the integral equation of the preceding section gives

$$= \int_{\partial\Omega} \mu_\alpha^* E \wedge (j_0 + \alpha j_1)$$

Taylor expansion of the field E on $\mathbb{R}^3$ around $\partial\Omega$

$$E(x + \alpha v(x)) = E(x) + \alpha(\mathcal{L}_v E)(x) + \dots$$

where $\mathcal{L}_v E$ is the Lie derivative along the vector field v. We get

$$\Delta S_\alpha = S_\alpha - S_0 = \alpha \int_{\partial\Omega} (\mathcal{L}_v E(x) \wedge j_0 + E \wedge j_1)$$

Covariance operator of the current distribution

We can use the relation $\mathcal{L}_v E = d(i_v E) + i_v(dE)$ from exterior differential analysis, together with the Maxwell equation $dE = -j\omega B$, to obtain

$$\Delta S_\alpha = \alpha \int_{\partial\Omega} (d(i_v E) \wedge j_0 + E \wedge j_1 - j\omega(i_v B) \wedge j_0)$$

$$\Delta S_\alpha = \alpha(\langle P, E\rangle + j\omega\langle M, B\rangle)$$

where we transposed the differential operator

$$\langle P, E\rangle = \int_{\partial\Omega} ((dj_0) i_v E - j_1 \wedge E) \quad \langle M, B\rangle = \int_{\partial\Omega} (j_0 \wedge i_v B)$$

We get the covariance operator (returning to N parameters)

$$C_J(F_1, F_2) = \sum_{k=1}^{N} \mathrm{var}(\alpha_k) \|(\langle P_k, E_1\rangle + j\omega\langle M_k, B_1\rangle)\|^2$$

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Analysis of B_1

The kernel of B_1 is essentially a function of the distance between two points on $\partial\Omega_\alpha$ expressed in coordinates on $\partial\Omega$

$$R_\alpha(x,y) \asymp \|x-y\| + \alpha\frac{(x-y)\cdot(v(x)-v(y))}{\|x-y\|} + O(\alpha^2)$$

where $\theta(x,y) = (x-y)/\|x-y\|$. For the free-space Green function we get up to first order

$$G_\alpha(x,y) \asymp G_0(x,y) - \alpha\theta(x,y)\cdot(v(x)-v(y))$$
$$(jk + 1/\|x-y\|)G_0(x,y) + O(\alpha^2)$$

The kernel distribution of B_1 is therefore given by the usual derivatives of the EFIE operator but with the modified Green function

$$G_1(x,y) = -G_0(x,y)\frac{\theta(x,y)\cdot(v(x)-v(y))(1+jk\|x-y\|)}{\|x-y\|}$$

A theoretical result

We have the following result: For normal deformations of a flat surface

$$\theta(x, y) \cdot (v(x) - v(y)) \equiv 0 \Rightarrow B_1 \equiv 0$$

and therefore $B_0(j_0 + \alpha j_1) = E_0 + \alpha E_1$ we call the result a "rubber sheet" deformation of a solution of the nominal integral equation

For normal deformations of locally flat surfaces the "rubber sheet" deformation of the current distribution is a locally correct first order approximation.

This could imply that the surface current distributions induced on a stochastic normal deformation of a surface, which is locally flat, on the scale of the correlation length, can be estimated by solving only conventional integral equations.

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Conclusion

In this talk, we have presented

- A general framework for the probabilistic approach to variability analysis of electromagnetic observable models based on canonical stochastic fields and stochastic boundary distributions
- An explicit perturbative construction of the covariance operator of a current distribution using boundary integral equations

Ongoing work at Onera is concentrating on non-perturbative computation of the covariance operator and higher-order statistics of observables.